Chapter 12.

Proof.

In Unit One of this Mathematics Specialist course we met the idea of *proof*.
In particular
- we deduced a number of geometrical truths by reasoning from other accepted truths, i.e. we used *deductive proof*,
- we used our understanding of *vectors* to prove a number of geometrical truths,
- we used *proof by contradiction*, in which the technique is to assume that the opposite of what we are trying to prove is true and then follow correct logical argument only to arrive at a contradiction, thus showing that our initial assumption must be wrong.

In this chapter we will continue our consideration of proof but now our emphasis is not so much on proving geometrical truths but instead we concentrate more on proving various truths involving real numbers, $\mathbb{R}$. The methods of *proof by exhaustion* and *proof by induction* are then particularly useful.
Real numbers are either rational (can be expressed as a fraction) or irrational (cannot be expressed as a fraction). To define rational and irrational numbers more formally we would say that rational numbers can be expressed in the form $\frac{a}{b}$ where a and b are integers with $b \neq 0$, whilst irrational numbers cannot be expressed in this form. Every real number has a decimal equivalent. The decimal equivalents of rational numbers are either terminating decimals or recurring decimals.

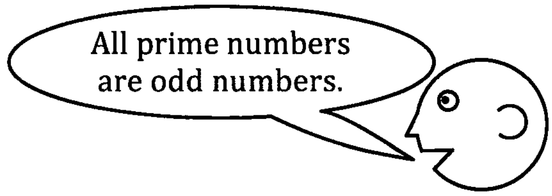
Example 1
Find the following recurring decimals as fractions: (a) 0·222 222 222 ...
(b) 0·212 121 212 ...

(a) Let $A = 0{\cdot}222\,222\,222\ldots$ ← ①
then $10A = 2{\cdot}222\,222\,222\ldots$ ← ②
② – ① $9A = 2$
Hence $A = \frac{2}{9}$

(b) Let $B = 0{\cdot}212\,121\,212\ldots$ ← ③
then $100B = 21{\cdot}212\,121\,212\ldots$ ← ④
④ – ③ $99A = 21$
Hence $A = \frac{21}{99}$ i.e. $\frac{7}{33}$.

You should also be familiar with the idea that one *counter example* can show a general *conjecture* to be false.

Consider, for example, the claim:

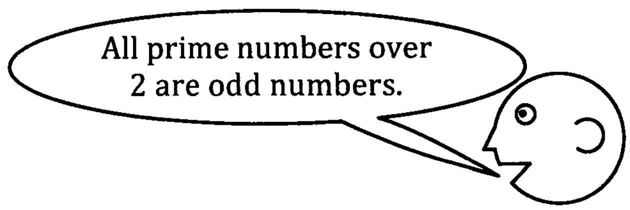


Checking some prime numbers:

13	-	an odd number
11	-	an odd number
7	-	an odd number
23	-	an odd number

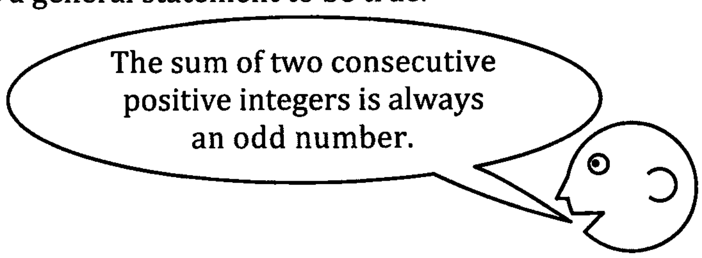
might lead us to believe the statement to be true but with just one counter example, the number 2, a prime number but not an odd number, we show the general statement to be false.

We might then adjust the statement in the light of the counter example:



In some cases we may be able to prove a general statement to be true.

Consider, for example, the claim:



Considering some specific cases:

For the consecutive positive integers 5 and 6:	$5 + 6 = 11$,	an odd number.
For the consecutive positive integers 12 and 13:	$12 + 13 = 25$,	an odd number.
For the consecutive positive integers 21 and 22:	$21 + 22 = 43$,	an odd number.

To prove the statement true we could proceed as follows:

If x is a positive integer then we can represent two consecutive positive integers as x and $x + 1$.
The sum of these two integers is then $\quad x + x + 1 = 2x + 1.$
Now with x an integer $2x$ must be even.
Hence $2x + 1$ must be odd and the statement is proved to be true.

Exercise 12A

For questions 1 to 10 state whether you think the given conjecture is true or false.
If you think it is false, give one example of when it is false.
If you think it is true, give three examples of when it is true, and try to prove it to be true.

1. If we square any even counting number greater than 2 and then subtract 1 we get a mutiple of 5.

2. The cube of any even integer is always a multiple of 8.

3. All multiples of 5 are also multiples of 10.

4. All right triangles are isosceles.

5. If we add together an integer squared, six times the integer and 9 we get a square number.

6. The sum of three consecutive positive integers will always be a multiple of 3.

7. The product of two even numbers is always even.

8. The square of an odd number is always an odd number.

9. The product of two consecutive even whole numbers is always a multiple of 8.

10. Multiplying any odd counting number by itself and then adding 7 always gives a multiple of 8.

11. Express each of the following numbers, each of which involve a recurring decimal, as a fraction.
    (a) 0·555 555 555 ...
    (b) $0\cdot\overline{75}$
    (c) 0·636 363 636 ...
    (d) $2\cdot\overline{231}$
    (e) 0·231 444 444 ...

12. By assuming that $\sqrt{2} = \frac{a}{b}$, a fraction expressed with a and b having no common factors (i.e. fully cancelled) and with a and b as integers, b ≠ 0, use the method of proof by contradiction to prove that $\sqrt{2}$ is in fact irrational.

Proof by exhaustion.

In this sense the word exhaustion is not used to mean that the proof tires us out and makes us exhausted! Instead the use of the word exhaustion means that the proof "exhausts all possibilities", it "considers completely all possible options". For example consider the following claim:

The square of any integer is always either a multiple of 5
or 1 more than, or 4 more than, a multiple of 5.

Now the integer to be squared could be

	a multiple of 5 itself.	Which we could represent as $5x$ for integer x.
	1 more than a multiple of 5.	Represented by $5x + 1$, for integer x.
	2 more than a multiple of 5.	Represented by $5x + 2$, for integer x.
	3 more than a multiple of 5.	Represented by $5x + 3$, for integer x.
or	4 more than a multiple of 5.	Represented by $5x + 4$, for integer x.

These possibilities together exhaust all options. Hence if we can prove the statement true for all these options we will have proved the statement true for all integers. Completing this proof is one of the questions of the next exercise.

Exercise 12B. Use proof by exhaustion for each of the following.

1. Prove that:
   The square of any integer always has the same parity as the integer.
   (The parity of a number refers to it being even or odd.)

2. Prove that:
   The square of any integer is always either a multiple of 5
   or 1 or 4 more than a multiple of 5.
   (Hint: See earlier on this page.)

3. By considering integers as multiples of 3
   or 1 more than a multiple of 3
   or ... ,

   Prove that:
   The cube of any integer is always either a multiple of 9
   or 1 more or 1 less than a multiple of 9.

4. A family of sequences is defined by the rule

   $$T_{n+1} = 3T_n + 2, \text{ where } T_n \text{ is the n}^{\text{th}} \text{ term.}$$

   For example,

   with $T_1 = 3$, $T_2 = 3(3) + 2 = 11$, $T_3 = 3(11) + 2 = 35$, $T_4 = 3(35) + 2 = 107$

   with $T_1 = 4$, $T_2 = 3(4) + 2 = 14$, $T_3 = 3(14) + 2 = 44$, $T_4 = 3(44) + 2 = 134$

   Prove that for sequences in this family, whatever the parity of a particular term is then the next term will have the same parity. (The parity of a number refers to it being even or odd.)

5. Prove that:
   For integer x, $x > 1$,

   $$x^5 - x$$

   is always a multiple of 5.

   factor($x^5 - x$)
   $x \cdot (x - 1) \cdot (x + 1) \cdot (x^2 + 1)$

   Is it always a multiple of 10?
   Is it always a multiple of 20? Justify your answers.

6. Prove that:
   For integer x, $x > 1$,

   $$x^7 - x$$

   is always a multiple of 7.

   factor($x^7 - x$)
   $x \cdot (x - 1) \cdot (x + 1) \cdot (x^2 + x + 1)(x^2 - x + 1)$

7. Noticing that

   $$3^3 - 3 = 24$$
   $$4^3 - 4 = 60$$
   $$5^3 - 5 = 120$$

   John conjectured (suggested) that

   For x any integer greater than 2 the expression $x^3 - x$ is always divisible by 12.

   Is John's conjecture correct?
   If yes, prove it. If no, make a similar conjecture of your own involving the divisibility of $x^3 - x$ and prove your conjecture true.

Proof by induction.

Consider the following sums of square numbers:

1^2	=	1	=	1
$1^2 + 2^2$	=	$1 + 4$	=	5
$1^2 + 2^2 + 3^2$	=	$1 + 4 + 9$	=	14
$1^2 + 2^2 + 3^2 + 4^2$	=	$1 + 4 + 9 + 16$	=	30
$1^2 + 2^2 + 3^2 + 4^2 + 5^2$	=	$1 + 4 + 9 + 16 + 25$	=	55
$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$	=	$1 + 4 + 9 + 16 + 25 + 36$	=	91

Verify that for each of the above the following formula is true:

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n}{6}(n+1)(2n+1)$$

Consider the following:

1×2	=	2	=	2
$1 \times 2 + 2 \times 3$	=	$2 + 6$	=	8
$1 \times 2 + 2 \times 3 + 3 \times 4$	=	$2 + 6 + 12$	=	20
$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5$	=	$2 + 6 + 12 + 20$	=	40
$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + 5 \times 6$	=	$2 + 6 + 12 + 20 + 30$	=	70

Verify that for each of the above the following formula is true:

$$1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n+1) = \frac{n}{3}(n+1)(n+2)$$

The previous page involved two rules,

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n}{6}(n+1)(2n+1)$$

and $$1 \times 2 + 2 \times 3 + 3 \times 4 + ... + n(n+1) = \frac{n}{3}(n+1)(n+2).$$

We could verify the rules to be true for various positive values of n but how would we **prove** the above formulae true for **all** positive integer values of n?

One suitable method of proof for these situations is **proof by induction**.

In proof by induction we follow the steps:

① Prove that **if** the statement is true for some general value of n, say n = k, then it must also be true for the next value of n, i.e. n = k + 1.

② Prove that there is a value of n, usually n = 1, for which the statement is true.

Question: Why do these two steps form a proof?

Answer: Step ② proves that the rule is true for n = 1 but then, by step ①, it must therefore be true for n = 2.

But if it is true for n = 2, step ① means that it must be true for n = 3.

But if it is true for n = 3, step ① means that it must be true for n = 4.

But if .... etc, etc.

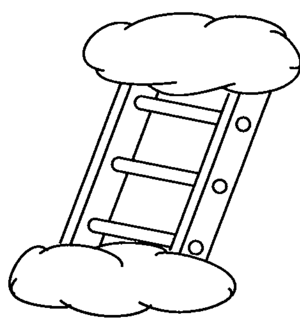
Hence the statement must be true for all positive integer n.

Proof by induction is like "an infinite ladder".

If we can prove that
- if any rung exists then the next rung must also exist,

and that
- at least one rung does exist,

then the infinite ladder must exist.



Example 2

Use the method of proof by induction to prove that

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$$

for all integer n ≥ 1.

Let us assume that the rule applies for n = k, i.e.

$$1^2 + 2^2 + 3^2 + \ldots\ldots + k^2 = \frac{k}{6}(k + 1)(2k + 1).$$

Now consider the situation for n = k + 1, i.e. consider

$$1^2 + 2^2 + 3^2 + \ldots\ldots + k^2 + (k + 1)^2$$

It follows that

$$\begin{aligned}1^2 + 2^2 + 3^2 + \ldots\ldots + k^2 + (k + 1)^2 &= \frac{k}{6}(k + 1)(2k + 1) + (k + 1)^2 \\ &= \frac{k+1}{6}\left(k(2k + 1) + 6(k + 1)\right) \\ &= \frac{k+1}{6}(2k^2 + 7k + 6) \\ &= \frac{k+1}{6}(k + 2)(2k + 3)\end{aligned}$$

Thus $\quad 1^2 + 2^2 + 3^2 + \ldots\ldots + (k + 1)^2 = \frac{k+1}{6}(k + 1 + 1)(2(k + 1) + 1)$

i.e. the initial rule applied for n = k + 1.

Hence, if the initial rule is true for n = k, it is also true for n = k + 1.

If n = 1, the rule claims that $\quad 1^2 = \frac{1}{6}(2)(3)$

$= 1$ which is true.

Thus: If the initial rule is true for n = k, it is also true for n = k + 1.

And: The rule is true for n = 1.

Hence, by induction, $1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$ for all integer n ≥ 1.

Exercise 12C

1. Use proof by induction to prove that

$$1 + 2 + 3 + 4 \;....\; n = \frac{1}{2}n(n+1)$$

for all integer n ≥ 1.

2. Prove, by induction, that

$$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ... + n(n+1) = \frac{n}{3}(n+1)(n+2)$$

for all integer n ≥ 1.

3. Prove, by induction, that

$$2 + 4 + 8 + 16 + 32 + ... + 2^n = 2^{n+1} - 2$$

for all integer n ≥ 1.

4. Use proof by induction to prove that

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + ... + n^3 = \frac{n^2}{4}(n+1)^2$$

for all integer n ≥ 1.

5. (a) Verify that the statements

$$\begin{aligned} 1 + 3 &= 4 \\ 1 + 3 + 5 &= 9 \\ 1 + 3 + 5 + 7 &= 16 \\ 1 + 3 + 5 + 7 + 9 &= 25 \end{aligned}$$

are consistent with the rule

$$1 + 3 + 5 + 7 + ... + (2n-1) = n^2.$$

(b) Use the method of proof by induction to prove the above rule to be true for all integer n ≥ 1.

6. Use proof by induction to prove that

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + ... + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$

for all integer n ≥ 1.

7. Use proof by induction to prove that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all integer n ≥ 1.

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8. Prove, by induction, that

$$1 \times 3 \times 5 + 2 \times 4 \times 6 + ... + n(n+2)(n+4) = \frac{n}{4}(n+1)(n+4)(n+5)$$

for all integer $n \geq 1$

9. Use proof by induction to prove that $(x - 1)$ is a factor of

$$x^n - 1$$

for all positive integer values of n.

10. Use proof by induction to prove that

$$1 \times 2 \times 3 \times 4 \times 5 \times 6 \; .... \times n \geq 3^n$$

for all integer values of $n > 6$.

11. Use the method of proof by induction to prove that

$$7^n + 2 \times 13^n$$

is a multiple of three for all $n \geq 0$.

12. Prove, by induction, that

$$2 - 4 + 8 - 16 + 32 \; .... \; (-1)^{n+1}\, 2^n = \frac{2}{3}\left(1 + (-1)^{n+1}\, 2^n\right)$$

for all integer $n \geq 1$.

Note.

Many questions in the previous exercise involved expressions like

$$1 + 2 + 3 + 4 + 5 + ...$$

$$1 + 3 + 5 + 7 + 9 + ...$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + ...$$

A shorthand way of writing $1 + 2 + 3 + 4 + 5 + 6 + 7$ is $\sum_{i=1}^{7} i$.

This is read as "sum all the i values starting from i = 1 and finishing at i = 7", (where i takes integer values).

Using this, **summation notation**, question 4, for example, could be written:

Prove, by induction, that $\sum_{i=1}^{n} i^3 = \frac{n^2}{4}(n+1)^2$

Extension Activity: Investigating some conjectures.

Do you understand the difference between a **conjecture** and a **theorem**?

If, based on our opinion or perhaps some observations or maybe some research, we think something to be true we might make a conjecture suggesting it as a truth. A conjecture could be our "best guess" at what seems to be the case. It may be based on incomplete information and has not been proven. Such a conjecture may later be proved to be true, in which case it would then become a theorem. On the other hand, perhaps someone, or some event, may prove the conjecture to be false.

You may convince others into believing your conjecture is true even though no proof is forthcoming. Just because a conjecture has not been proven true it may also not have been proven false and may be considered by all to be a truth, even though unproven. A theorem on the other hand is a statement that has been proved to be true, often by reasoning from other known truths.

Investigate each of the following famous conjectures. What does each conjecture claim? Give some examples of what it is claiming to be the case. What is the history of the conjecture? Who made the conjecture? When? Where? Has it since been proven to be true, or perhaps false? Etc.
Write a report about each conjecture.

Goldbach's conjecture.

The twin prime conjecture.

Fermat's conjecture.

The four colour conjecture.

Miscellaneous Exercise Twelve.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

1. If $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ determine each of the following. If any cannot be determined state this clearly.

   (a) AB (b) BA (c) BC (d) CD (e) BD

2. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$ determine matrices B, C, D and E given that

   $AB = \begin{bmatrix} 13 \\ -4 \end{bmatrix}$, $AC = \begin{bmatrix} 13 \\ 6 \end{bmatrix}$, $DA = \begin{bmatrix} 6 & 19 \end{bmatrix}$, and $EA = \begin{bmatrix} 5 & 0 \end{bmatrix}$.

3. If $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 21 \\ 9 & 17 \end{bmatrix}$ and AC = B, find C.

4. In the first copy of a new magazine for "would be stamp collectors" an invitation is made to each purchaser of the magazine to complete a six month subscription order and receive a bonus "free starter pack". Two types of pack are available with the contents of each as shown below.

	Number of Australian stamps	Number of Rest of the world stamps
Each *Mainly Australian* starter pack:	75	25
Each *Rest of the World* starter pack:	20	80

   We will call this matrix X.

   The offer prompts 210 requests for the *Mainly Australian* starter pack and 120 requests for the *Rest of the World* starter pack.

   We could write this as a column matrix, Y: $\begin{bmatrix} 210 \\ 120 \end{bmatrix}$

   or as a row matrix, Z: $\begin{bmatrix} 210 & 120 \end{bmatrix}$

   (a) Which of the following matrix products could be formed:

   XY, YX, XZ, ZX?

   (b) Of those matrix products in (a) that can be formed, which will contain information that is likely to be of use?

   (c) Determine the useful products from (b) and explain the information displayed.

5. Given that $A = \begin{bmatrix} x & 1 \\ 0 & 3 \end{bmatrix}$ and $A^2 + A = \begin{bmatrix} 6 & x^2 - 8 \\ p & q \end{bmatrix}$ determine p, q and x.

6. Prove that $\sin 2\theta = \dfrac{2 \tan \theta}{\tan^2 \theta + 1}$

7. Prove that $\sin 5x \cos 3x - \cos 6x \sin 2x = \sin 3x \cos x$.

8. (a) Express $(5 \cos \theta - 3 \sin \theta)$ in the form $R \cos (\theta + \alpha)$ for α an acute angle in radians and correct to two decimal places.
   (b) Hence determine the minimum value of $(5 \cos \theta - 3 \sin \theta)$ and the smallest positive value of θ (in radians and correct to two decimal places) for which it occurs.

9. The matrices A, B and C shown below can be multiplied together to form a single matrix if A, B and C are placed in an appropriate order. What is the order and what is the single matrix this order produces?

$$A = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}.$$

10. If $A = \begin{bmatrix} 2x & x \\ 4 & y \end{bmatrix}$ and $A^2 = \begin{bmatrix} 24 & p \\ 0 & q \end{bmatrix}$ find all possible values of x, y, p and q.

11. If AB = AC, A ≠ O, then matrix B does not necessarily equal matrix C, as the following examples show:

Example 1: $A = \begin{bmatrix} 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}.$

$$AB = \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 7 & -4 \end{bmatrix} \quad AC = \begin{bmatrix} 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 7 & -4 \end{bmatrix}$$

Thus AB = AC, A ≠ O, but B ≠ C.

Example 2: $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix}.$

$$AB = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 7 & 4 \end{bmatrix} \quad AC = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 7 & 4 \end{bmatrix}$$

Thus AB = AC, A ≠ O, but B ≠ C.

Do the examples above conflict with the following proof that if AB = AC then B = C?

If	AB	= AC
then	$A^{-1}AB$	$= A^{-1}AC$
	IB	= IC
and so	B	= C

12. BC is just one product that can be formed using two matrices selected from the four below. List all the other products that could be formed in this way. (The selection of the two matrices can involve the same matrix being selected twice.)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}.$$

13. Triangle ABC has vertices A(2, 0), B(2, 3) and C(4, 3). Find the coordinates of the vertices of triangle A'B'C', the image of ABC when transformed using the transformation matrix $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.
Show both ΔABC and ΔA'B'C' on grid paper.
What is the transformation this matrix represents?

14. Prove that

$$\sec x \ \operatorname{cosec} x \ \cot x = 1 + \cot^2 x$$

15. Find all solutions to the equation

$$7 \sin x + \cos x = 5$$

rounding answers to two decimal places when rounding is appropriate.

16. Prove, by induction, that

$$12 + 19 + 31 + 53 + ... + [\,5(1 + 2^{n-1}) + 2n\,] = n(n + 6) + 5(2^n - 1)$$

for all integer n ≥ 1.

17. Prove by induction that $3^{2n+4} - 2^{2n}$ is divisible by 5 for all positive integer *n*.

18. Prove that $5^n + 7 \times 13^n$ is a multiple of 8 for all integer n ≥ 1.

19. Prove, by induction, that for r ≠ 1 and all integer n ≥ 1,

$$r + r^2 + r^3 + r^4 + ... + r^n = \frac{r(r^n - 1)}{r - 1}$$